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LETTER TO THE EDITOR

Algebraic aspects of quantum statistics

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Abstract. Consistent implementation of the additivity of physical observables in second quantization uniquely establishes quantum statistics. In this scheme the usual Weyl–Heisenberg algebra induces Maxwell–Boltzmann distribution, while Bose–Einstein statistics are related to $su(1,1)$.

Isomorphism of physical observables with operators of the universal enveloping algebra and representation theory are the best known applications of Lie algebras. In first quantization the whole Lie–Hopf structure [1] has been used, even though often without explicitly realizing it. Indeed it is the Lie–Hopf structure which prescribes that observables such as energy, momentum and angular momentum must be additive. Without this property we should loose the essential notion of an isolated system.

In this paper we aim to prove that the full Hopf algebra, and not the commutation relations only, must be taken into account for second quantization as well, both for physical reasons (for instance the additivity implied by the co-algebra is necessary to equally occupy the states in a microcanonical ensemble) and because Hopf algebras qualify as the natural mathematical tool to describe observables on a direct-product space.

Our main result is that in analogy with, for example, additivity of angular momenta, which fixes by means of the Clebsch–Gordan coupling scheme the rules to compose two spins, additivity of creation and annihilation operators determines quantum statistics, and the latter is not always identical with that customarily adopted. We shall show that the algebra naturally pertaining to Bose statistics in this perspective is $su(1,1)$, whereas the Heisenberg algebra $h(1)$ is appropriate to the Maxwell–Boltzmann statistics.

Creation and annihilation operators $a_\ell^\dagger$ and a_ℓ in second quantization are nothing but ladder operators, whose function is to generate the whole accessible Fock space of states $\mathcal{F} = \mathrm{span}\{|n_1, n_2, \ldots\rangle\}$. These operators are assumed independent for different ℓ's: the traditional procedure to describe bosons consists in adopting for them the Weyl–Heisenberg algebra $h(1)$. Their only duty, however, is to connect the n-particle Fock space sector $\mathcal{F}_n$ to $\mathcal{F}_{n\pm1}$ ($\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n$), which have no memory of the algebra, nor have they any intrinsic statistics which has instead to be imposed 'by hand' as external constraints. In order to check this, we refer to the pedagogical discussion given by Roman in [2], based on a two-particle system, each particle living in one of only two possible states. In addition to fermions, which obey the Pauli exclusion principle, Roman points out two cases with unlimited occupation numbers: boson statistics of course, but also Maxwell–Boltzman statistics. In this latter classical distribution, although one still deals with quantum particles, the fundamental equiprobable event is equivalent to putting a 'marble' into one of two 'boxes': this implies that the distribution with the two particles in different states has weight $\frac{1}{2}$ while the two distributions with both particles in the same state each have weight $\frac{1}{4}$. In Bose–Einstein statistics, in contrast, the fundamental equiprobable objects are the vectors of $\mathcal{F}|2,0\rangle$, $|1,1\rangle$ and $|0,2\rangle$, all of which have the same weight $\frac{1}{3}$. Such weights are the constraints, independent from the algebra, with which the Fock space is to be equipped.

In our approach the statistical constraints derive from the same Lie–Hopf algebra of raising and lowering operators, $\mathcal{A}$. We shall show that each representation in $\mathcal{F}$ of $\mathcal{A}$ satisfying the physical request of the existence of a vacuum defines a quantum statistics, and that to the standard quantum statistics corresponds the fundamental representation of a Lie–Hopf algebra. We have in fact singled out four algebras, thus finding further quantum statistics. In order of increasing 'attractivity' [3]:

(i) Fermi–Dirac statistics, related to the superalgebra $h(1|1)$, completely repulsive (because a fermion forbids that a new one occupies the same quantum state), which we shall not discuss;

(ii) Maxwell–Boltzman statistics, related to the Heisenberg algebra $h(1)$, completely neutral (a new particle 'chooses' its state independently from the others);

(iii) Bose–Einstein statistics, moderately attractive (where all vectors in the Fock space have the same weight) that we shall show to be related to the fundamental representation of $su(1,1)$; other representations give rise to different interpolating statistics;

(iv) new strongly attractive statistics, related to the superalgebra $osp(1|2)$, where the second particle is forced to occupy the same state as the first.

The core of our argument is that to each Lie algebra $\mathcal{A}$ is associated a Lie–Hopf algebra [1]; namely $\mathcal{A}$ is equipped, besides the usual commutation relations, with three additional maps: *coproduct*, *co-unit*, *antipode*. Among these, the coproduct $\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ identifies the co-algebra related to the direct-product representation.

The physical meaning of Δ is understood if one thinks of the process of addition of two angular momenta: $J^\alpha_{\rm tot} \equiv J^\alpha_1 + J^\alpha_2, \alpha = 1, 2, 3$. The requirement that the components of $J^\alpha_{\rm tot}$ still define an angular momentum operator implies the well known Clebsch–Gordan coupling scheme. It also implies that $J^\alpha_{\rm tot}$, written more formally as $\Delta(J^\alpha) = J^\alpha \otimes 1 + 1 \otimes J^\alpha$, is nothing but the co-algebra of the Lie–Hopf algebra $su(2)$. More generally, in the operations, quite common in physics, of disassembling a complex system into its separate components as well as of collecting together non-interacting subsystems into a single object, energy, momentum and other observables are implicitly considered as additive. We argue that, analogously, in second quantization creation and annihilation operators must be additive if one wants to impose, for example, that two degenerate levels are occupied with equal probability as happens e.g. in the microcanonical ensemble. This means not only that the global algebra must be isomorphic with that of the individual components, which is a fundamental property of all Hopf algebras, but that one must have an additive algebra, i.e. a Lie–Hopf algebra (an example of an algebra with non-additive co-algebra is given at the end of the paper). This is rather more than what is contained solely in the commutation relations and requires the information contained in the coproduct. One has thus constraints for constructing pure states, as opposed to mixed states.

For simplicity, we confine our attention to the two-mode case, studying states of the form $|k, n-k\rangle$, $k = 0, 1, \ldots, n$; generalization to the case of m modes, $m > 2$, is straightforward. The main point of our argument consists of the way of constructing the set of such two-mode states $\{|\phi_n\rangle\}$, automatically incorporating the statistical constraints. We do this by properly composing first creation (raising) operators by the coproduct

$$\Delta(a^\dagger) = a^\dagger \otimes 1 + 1 \otimes a^\dagger \equiv a_1^\dagger + a_2^\dagger$$

and successively acting on the two-mode vacuum $|0,0\rangle$ ($\Delta(N)|0,0\rangle = 0 = \Delta(a)|0,0\rangle$) to obtain

$$\{|\phi_n\rangle \propto (\Delta(a^\dagger))^n|0,0\rangle\}.$$

In such a framework we shall consider the algebras $h(1)$, $osp(1|2)$ and $su(1,1)$. For simplicity we shall adopt for all of them the same notation for the generators.

$h(1)$ has generators a, $a^\dagger$ and N, as well as the identity $\mathbf{I}$, which is central, with commutation relations

$$[a, a^\dagger] = \mathbf{I} \qquad [N, a] = -a \qquad [N, a^\dagger] = a^\dagger.$$

We assume that $\mathbf{I} \simeq 1$ in the representation. The requirement that $a|0\rangle = 0$ (i.e. the physical vacuum $|0\rangle$ is the highest weight vector) implies $N \simeq a^\dagger a$, which is customarily adopted in the description of bosons.

$osp(1|2)$ is a $\mathbb{Z}_2$-graded algebra. Its universal enveloping algebra is generated by a, $a^\dagger$ and H, with the relations

$$\{a, a^\dagger\} = 2H \qquad [H, a] = -a \qquad [H, a^\dagger] = a^\dagger.$$

With the position $H \equiv N + \frac{1}{2}$, dictated by the interpretation of N as number of 'quanta', the Fock space representation for $osp(1|2)$ is the same:

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad N|n\rangle = n|n\rangle.$$

More details on $osp(1|2)$ can be found in [4, 5]; we stress here only that, since $a^\dagger$ is an odd operator, we have $\{a_i^\dagger, a_j^\dagger\} = 0$, for $i \neq j$ (in contrast to the other cases, in which they commute), which imply that $osp(1|2)$ may be related, in some way, also to fermions, in analogy with its contraction $h(1|1)$.

$su(1,1)$ has generators a, $a^\dagger$ and H:

$$[a, a^\dagger] = 2H \qquad [H, a] = -a \qquad [H, a^\dagger] = a^\dagger.$$

Again $H \equiv N + \frac{1}{2}$ and we select the fundamental representation $D^+_{1/2}$ [6], written, in the same Fock space,

$$a|n\rangle = n|n-1\rangle \qquad a^\dagger|n\rangle = (n+1)|n+1\rangle \qquad N|n\rangle = n|n\rangle.$$

The algebraic structures considered above straightforwardly lead to the normalized two-mode states:

$$|\phi_n\rangle = 2^{-n/2}\sum_{k=0}^{n}\sqrt{\binom{n}{k}}|k, n-k\rangle \qquad \text{for } \mathcal{A} \sim h(1)$$

$$|\phi_n\rangle = \frac{1}{\sqrt{n+1}}\sum_{k=0}^{n}|k, n-k\rangle \qquad \text{for } \mathcal{A} \sim su(1,1)$$

while, for $\mathcal{A} \sim osp(1|2)$,

$$|\phi_{2n}\rangle = \frac{1}{2^n}\sum_{k=0}^{n}\sqrt{\binom{2k}{k}\binom{2(n-k)}{n-k}}|2k, 2(n-k)\rangle$$

$$|\phi_{2n+1}\rangle = \frac{1}{2^n\sqrt{n+1}}\sum_{k=0}^{n}\sqrt{\binom{2k}{k}\binom{2(n-k)}{n-k}}$$
$$\{\sqrt{n-k+1/2}\,|2k, 2(n-k)+1\rangle + \sqrt{k+1/2}|2k+1, 2(n-k)\rangle\}.$$

Because of orthonormality of states, phases are irrelevant so far as no interaction is switched on.

From $|\phi_n\rangle = \sum c_k^{(n)}|k, n-k\rangle$, one obtains in a straightforward way the normalized distribution function at fixed particle number n, $\mathcal{P}_n(k) = |c_k^{(n)}|^2$. In the case of $h(1)$ we find the binomial distribution with probability $p = \frac{1}{2}$:

$$\mathcal{P}_n(k) = \frac{1}{2^n}\binom{n}{k}. \tag{1}$$

Thus, $h(1)$ is naturally associated with the classical statistics of identical objects to be distributed with equal *a priori* probability in two identical slots (recovering, for $n = 2$, the example of Roman for the Maxwell–Boltzmann case). For $su(1,1)$ we have a probability distribution

$$\mathcal{P}_n(k) = \frac{1}{n+1} \qquad \forall k \tag{2}$$

uniform for all accessible states. This corresponds to the case of bosons (once more compare for $n = 2$ the example of Roman). Finally, $osp(1|2)$ gives rise to

$$\mathcal{P}_{2r+1}(2s) = \frac{1}{2^{2r}}\frac{r-s+\frac{1}{2}}{r+1}\binom{2s}{s}\binom{2(r-s)}{r-s} \qquad \mathcal{P}_{2r}(2s) = \frac{1}{2^{2r}}\binom{2s}{s}\binom{2(r-s)}{r-s}$$

$$\mathcal{P}_{2r+1}(2s+1) = \frac{1}{2^{2r}}\frac{s+\frac{1}{2}}{r+1}\binom{2s}{s}\binom{2(r-s)}{r-s} \qquad \mathcal{P}_{2r}(2s+1) = 0$$

which distinguish between the various possible combinations of parities (and suggest nuclear physics as a possible realm of application). These distributions have a more transparent meaning in the limit when the number of particles is infinite, $r \to \infty$. Upon introducing the variable $x \equiv s/r$ $(0 \leqslant x \leqslant 1)$ and defining $\lim_{r\to\infty}\mathcal{P}_{2r}(2s) = p_{\text{ee}}(x)$, $\lim_{r\to\infty}\mathcal{P}_{2r+1}(2s) = p_{\text{oe}}(x)$, $\lim_{r\to\infty}\mathcal{P}_{2r+1}(2s+1) = p_{\text{oo}}(x)$ (p_{eo} is of course identically zero), the above equations give the probability densities

$$p_{\text{ee}}(x) = \frac{2}{\pi}\frac{1}{\sqrt{x(1-x)}} \qquad p_{\text{oe}}(x) = \frac{2}{\pi}\sqrt{\frac{1-x}{x}} \qquad p_{\text{oo}}(x) = \frac{2}{\pi}\sqrt{\frac{1-x}{x}}$$

which exhibit at the extremes integrable divergences signalling maximum attractivity in the sense of Huang [3].

The entire procedure described so far holds, of course, for any Hopf algebraic structure, whether or not it is a Lie algebra, because the coproduct map Δ is well defined. Thus the whole construction can be repeated for the q-algebras associated with the three algebras utilized above. We report here the result concerning $h_q(1)$, which is a clear example that Lie algebra does not fix the full structure.

$h_q(1)$ [7] is the deformation of the universal enveloping algebra of $h(1)$ controlled by the arbitrary parameter q:

$$[a, a^\dagger] = \frac{q^{2\mathbf{I}} - q^{-2\mathbf{I}}}{q^2 - q^{-2}} \simeq 1 \qquad [N, a] = -a \qquad [N, a^\dagger] = a^\dagger \tag{3}$$

and with a coproduct map which for $a^\dagger$ is

$$\Delta(a^\dagger) = a^\dagger \otimes q^{\mathbf{I}} + q^{-\mathbf{I}} \otimes a^\dagger \simeq q a_1^\dagger + q^{-1} a_2^\dagger. \tag{4}$$

Its algebra equation (3) is manifestly isomorphic to $h(1)$, but its Clebsch–Gordan decomposition equation (4) is quite different:

$$|\phi_n\rangle = \frac{1}{(q^2+q^{-2})^{n/2}} \sum_{k=0}^{n} \sqrt{\binom{n}{k}} q^{2k-n} |k, n-k\rangle.$$

$$\mathcal{P}_n(k) = \frac{1}{(q^2+q^{-2})^n} \binom{n}{k} q^{4k-2n}$$

shows that $h_q(1)$ generalizes the binomial distribution to $p = (1+q^{-4})^{-1}$. Assuming for the two modes energies $E_1 > E_2$, the parameter q can then be related to temperature T: $q \to 0$ implies that only state 2 is filled (see (4)), hence $T = 0$, whereas for $q = 1$ both energy levels are statistically equally occupied and T must be infinite.

The cases of $su_q(1,1)$ and $osp_q(1|2)$ have some mathematical interest, and will be discussed elsewhere [8].

Summarizing, we have exhibited the close link between co-algebra and quantum statistics. The relevant ingredients of such a link are the following. Lie–Hopf algebras impose the additivity of generators and therefore of the related physical observables. Consistent use of such additivity in second quantization authomatically reduces the Fock space dimension according to the statistical constraints.

Besides $h(1|1)$, which we did not discuss, we have singled out three algebras. $osp(1|2)$ leads to novel strongly attractive statistics. $h(1)$ turns out to generate the Maxwell–Boltzmann probability distribution. $su(1,1)$ exhibits a much more articulated picture: the requirement of the existence of a vacuum imposes that one takes into consideration the discrete series bounded below D_k^+ [6]. The fundamental representation, $k = \frac{1}{2}$, originates Bose–Einstein statistics, the contraction $k \to \infty$ recovers $h(1)$ and hence Maxwell–Boltzmann, whereas the intemediate k's give rise to statistics in between.

We conclude with a few additional remarks. In the frame of a grand-canonical ensemble, the states have the (coherent) form $|\phi\rangle \propto \mathrm{e}^{\Delta(a^\dagger)}|0,0\ldots\rangle$. For the representation $D_{1/2}^+$ of $su(1,1)$ this leads to $\mathrm{e}^{\Delta(a^\dagger)}|0,0\ldots\rangle = \sum_{n_1,n_2,\ldots}|n_1,n_2\ldots\rangle$ which indeed gives the statistics introduced by Bose [9], whereas $h(1)$ uniquely implies $\mathrm{e}^{\Delta(a^\dagger)}|0,0\ldots\rangle = \sum_{n_1,n_2,\ldots}(n_1!n_2!\ldots)^{-1/2}|n_1,n_2\ldots\rangle$ which leads just to the Boltzmann statistics with the correct counting.

Our discussion is limited here to second quantization. The question of whether it can be extended to the continuum, giving rise to a corresponding field theory, is still under investigation. The commutations adopted should still imply for the field commutators a factor $\delta(x-y)$ also for $su(1,1)$—even though possibly weighted by a kernel—thus guaranteeing locality of the theory. Moreover, in the customary applications, quantum field theory characteristically deals with singly occupied bosonic states, a case where the representation $D_{1/2}^+$ of $su(1,1)$ and $h(1)$ (projected onto the Fock subspace with $n_j < 2$) give exactly the same results.

The method proposed here suggests instead the possibility of new interesting perspectives from the point of view of applications in statistical mechanics, where assigning a given algebra for a_ℓ and $a_\ell^\dagger$ implies—in our scheme—assigning the particle statistics. For example, it appears plausible that $h(1)$ may indeed play its most relevant role in systems where Maxwell–Boltzmann statistics may emerge naturally from a dynamics intrinsically quantum. Similarly we expect that mesoscopic properties related to the existence of a condensate are more properly connected with the $su(1,1)$ distribution. Finally, the novel distribution generated by $osp(1|2)$ has features which appear possibly related to superconductivity [8].

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